Malaysian Journal of Mathematical Sciences 19(3): 1133–1156 (2025)
https://doi.org/10.47836/mjms.19.3.18

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Fractional Epidemic Model of Tuberculosis Disease with Media Impact on The Migrants and Seasonal Farm Workers

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Received: 30 January 2025
Accepted: 15 April 2025

Abstract

The impact of media awareness can reduce the number of tuberculosis-infected individuals to a limited extent. This paper presents a non-integer mathematical model of tuberculosis disease using the Caputo operator. The primary objective of this study is to investigate the impact of media awareness on tuberculosis-infected migrants and seasonal farm workers. The qualitative analysis of the existence and uniqueness of the solutions, basic reproduction number $\mathbf{R_0}$, disease free equilibrium point, sensitivity analysis, and the Hyers-Ulam stability of the model are also examined. We provide numerical simulations to illustrate the model's behavior for various fractional orders. In our findings, $\mathbf{R_0} = 0.5473$ demonstrates that when more infected migrants and seasonal farm workers seek preventative treatment and awareness about tuberculosis disease, the infection rate will rapidly decline in the respective region.

Keywords: existence and uniqueness criteria; Hyers-Ulam stability; modified Euler method.

1 Introduction

The microorganisms that are harmful, like viruses, bacteria, fungi, and parasites, cause infectious diseases. These infections can be disseminated directly or indirectly between individuals or by other vectors such as insects or contaminated food and water. Infectious diseases have impacted human communities historically, ranging from minor outbreaks to worldwide pandemics [34, 14]. Mycobacterium tuberculosis (mTB) is the pathogen that causes tuberculosis (TB), also known as pulmonary tuberculosis, and usually affects the lungs. But occasionally, it can also have an impact on the kidneys, spinal cord, skeleton, etc. When a person with active TB coughs, sneezes, or spits, the disease can spread through the air [32]. Archaeological evidence from ancient Egypt, India, and China has revealed the existence of this ancient disease [27].

The mathematical model demonstrates a prominent application to study the development or decomposition of TB bacteria and how to control TB by adjusting biological parameters [7, 10]. The main purpose of mathematical modelling on infectious disease transmission is to provide insightful and pertinent perspectives for public health initiatives meant to stop or even slow the disease spread. Many aspects of TB's natural history and transmission dynamics, however, are still unclear because the disease is contagious and has a complicated mode of transmission. A tremendous deal of research has been done on the mathematical modelling and analysis of TB [37]. Numerous researchers have conducted substantial research on TB's mathematical modelling and analysis. To analyze the TB epidemiological pattern, Waaler et al. [35] developed the first mathematical model. Das et al. [13] analyzed a TB model incorporating the influence of media on transmission rate. It is important to consider various aspects of TB while modelling the disease mathematically. These include vaccination [29], therapy and incomplete treatment [26], quick and slow progression [9], reinfection [11], and drug-resistant strains [36].

Recently, the application of fractional-order derivatives in mathematical modelling has become somewhat well-known in many disciplines, including epidemiology [23] and control theory [17]. Fractional-order derivatives exhibit some key physical properties, such as memory effects, nonlocality, and flexibility. These properties make fractional derivatives valuable in various fields, including control theory, bioengineering, and complex systems modelling [22]. The memory effect, a crucial component of biological models, is one of the primary characteristics [4]. Many researchers studied the fractional order TB models to study the existence and uniqueness criteria [19], dynamics in two age groups [16], insufficient treatment [5], and Hyers-Ulam (HU) stability [24]. Al-Mdallal et al. [2] studied a fractional order coronavirus model via the modified Euler method (MEM). Furthermore, he used the Runge-Kutta fourth order (RK4) method step by step to investigate the coronavirus transmission dynamics [3].

Furthermore, advancements in TB control necessitate a comprehensive understanding of its cause, transmission dynamics, preventative strategies, treatment modalities, and a positive disposition. Addressing knowledge deficiencies in TB prevention is crucial for eradicating the disease [15]. Numerous researchers discussed some factors of disease knowledge and attitude, i.e., literacy [1], media [31], professional occupation [21], education about health [6], and cultural myths [18]. Since knowledge is fundamental to a TB control strategy, assessing the awareness of migrants and seasonal farm workers is essential for the effective and expedited implementation of TB control measures [20]. Despite considerable advancements in TB management and mitigation in the particular region, evidence is inadequate on the present knowledge, attitudes, and associated variables regarding TB among seasonal farm workers and migrants in the relevant region. As a result, assessing their knowledge and attitude is critical in taking action to reach the end TB strategy goal. This research aims to analyze the extensive details of the media impact on migrants and seasonal farm workers via the fractional order Caputo derivative operator with the

MEM method. Additionally, we use the RK4 method to compare and check the validity of the results. Furthermore, as preventive treatment is a voluntary tool for disease control, the media will have an effect on the rate of treatment adoption. The media can encourage various migrants and seasonal farm workers, particularly those who have low knowledge about TB disease, to seek preventative therapy by highlighting the advantages and necessity of doing so for those who are infected with it.

We structure this manuscript in the following way: In Section 2, we construct a novel fractional-order epidemic model that considers the impact of media awareness on TB-infected migrants and seasonal farm workers. Section 3 delineates the essential definitions of the Caputo operator. In addition, Section 4 examines the existence of the proposed model unique solution. Next, we compute the disease-free equilibrium point (DFEP) $\mathbf{E_0}$, basic reproduction number $\mathbf{R_0}$, and sensitivity analysis in Section 5. Section 6 highlights HU stability. In Section 7, we employ the MEM approach to derive an approximate solution for the proposed model. Furthermore, we examine the influence of characteristics associated with preventative treatment on TB control and validate the results using the RK4 approach. Ultimately, Section 8 delineates the conclusion.

2 Model Formulation

In this paper, we presented fractional differential equations under the Caputo derivative to examined the TB disease dynamics by dividing the population into six categories according to the population's epidemiological status: $S(\zeta)$ susceptible population, $E(\zeta)$ exposed population, $I(\zeta)$ infected population, $I_W(\zeta)$ migrants and seasonal farm workers infected population, $T(\zeta)$ treatment population, and $R(\zeta)$ recovered population. The overall population is denoted by $N(\zeta)$.

$$\begin{cases} {}^C D_0^\eta S(\zeta) = \Lambda - \beta S(I+\theta T) + \gamma R - \mu S, \\ {}^C D_0^\eta E(\zeta) = \beta S(I+\theta T) - \left(\alpha \dfrac{T}{m+T} + \omega + \mu\right) E, \\ {}^C D_0^\eta I(\zeta) = \omega E - (\delta + d_1 + \mu) I, \\ {}^C D_0^\eta I_W(\zeta) = \alpha \dfrac{T}{m+T} E - (\phi + r_2 + \mu) I_W, \\ {}^C D_0^\eta T(\zeta) = \delta I + \phi I_W - (r_1 + d_2 + \mu) T, \\ {}^C D_0^\eta R(\zeta) = r_1 T + r_2 I_W - (\gamma + \mu) R. \end{cases} \tag{1}$$

Figure 1 illustrates the schematic diagram of the above Model (1). The terms leaving the compartment are indicated by the outward arrow, while the terms entering the compartment are indicated by the inward arrow.

The following presumptions are used in the TB model: Let β represent the transmission rate and Λ indicate the recruitment in the susceptible population by migration and birth. The expression θ represents the reduction rate of TB transmission attributable to therapy, $\theta \in (0,1)$, γ signifies the recovery rate, which regains susceptibility after losing immunity, and μ represents natural death. The impact rate influenced by media is denoted by α, whereas m signifies the half-saturation constant. The expression ω is the ratio at which the exposed individuals become infected, and ϕ is the transmission ratio of infected workers from I_W into T. d_1 and d_2 represent the induced disease mortality ratio in I and T respectively. The parameter δ represents the transmission ratio of infected individuals to the T, and r_1 and r_2 are the recovery rates in T and I_W, respectively.

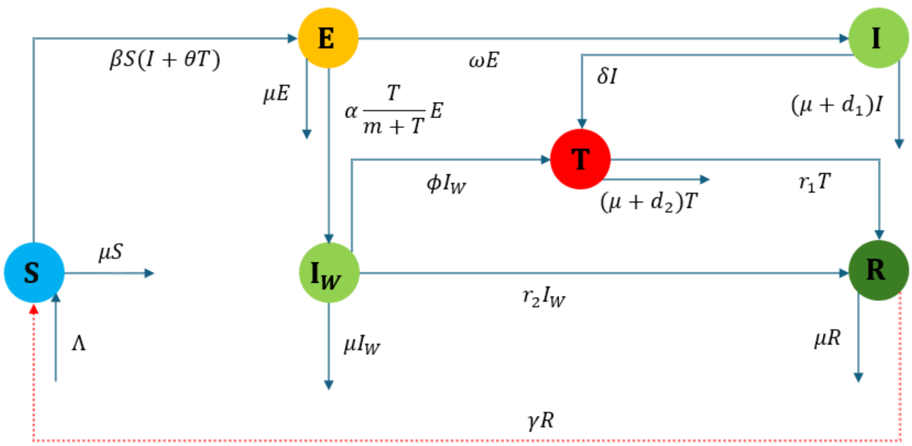

Figure 1: Schematic diagram of Model (1).

Note that all parameters are positive except $\gamma \geq 0$ and $\alpha \geq 0$. In this case, $\gamma = 0$ denotes permanent immunity for recovered individuals, whereas $\alpha = 0$ denotes no media impact. Let $k_1 = \omega + \mu$, $k_2 = \delta + d_1 + \mu$, $k_3 = \phi + r_2 + \mu$, $k_4 = r_1 + d_2 + \mu$, $k_5 = \gamma + \mu$. Model (1) can be written as,

$$\begin{cases} {}^C D_0^\eta S(\zeta) = \Lambda - \beta S(I+\theta T) + \gamma R - \mu S, \\ {}^C D_0^\eta E(\zeta) = \beta S(I+\theta T) - \alpha \dfrac{T}{m+T} E - k_1 E, \\ {}^C D_0^\eta I(\zeta) = \omega E - k_2 I, \\ {}^C D_0^\eta I_W(\zeta) = \alpha \dfrac{T}{m+T} E - k_3 I_W, \\ {}^C D_0^\eta T(\zeta) = \delta I + \phi I_W - k_4 T, \\ {}^C D_0^\eta R(\zeta) = r_1 T + r_2 I_W - k_5 R, \end{cases} \tag{2}$$

subjected to the initial conditions,

$$(S, E, I, I_W, T, R) \geq 0.$$

In the above model, the fractional order derivatives denoted by ${}^C D_0^\eta$, where $0 < \eta \leq 1$ are the Caputo derivatives associated with biological parameters.

3 Caputo Operator Basic Definitions

Definition 3.1. [30] *For any function $f(\zeta)$, the Caputo derivative of order η is defined as follows,*

$${}^C D_0^\eta f(\zeta) = \frac{1}{\Gamma(n-\eta)} \int_0^\zeta (\zeta - s)^{n-\eta-1} f^{(n)}(s) ds, \tag{3}$$

where $n = [\eta] + 1$, the Laplace transformation is defined as,

$$L\{{}^C D_0^\eta f(\zeta)\} = s^\eta F(s) - \sum_{k=0}^{n-1} s^{(\eta-k-1)} f^{(k)}(0). \tag{4}$$

Definition 3.2. [33] *The Mittag-Leffler function of two parameters in series is defined as,*

$$E_{\eta,q}(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(\eta k + q)}, \qquad \eta, q \in R, \qquad \zeta \in C. \tag{5}$$

The Laplace transformation is defined as,

$$L\left\{\zeta^{q-1} E_{\eta,q}(\pm M\zeta^{\eta})\right\} = \frac{s^{\eta-q}}{s^{\eta} \mp M}. \tag{6}$$

4 Existence & Uniqueness of Solution

We demonstrate the existence and uniqueness of Model (2) via fixed point theory. Let $\mathcal{Q}(\zeta)$ and $\mathcal{Q}(0)$ be two vectors, which have state variables with initial values, and a vector function $\mathcal{Y} : [0, T] \times R^6 \to R$ is continuous, i.e.,

$$\begin{cases} \mathcal{Q}(\zeta) = (S, E, I, I_W, T, R)^T, \\ \mathcal{Q}(0) = (S_0, E_0, I_0, I_{W0}, T_0, R_0)^T, \\ \mathcal{Y}(\zeta, \mathcal{Q}(\zeta)) = (\mathcal{B}_i)^T, \qquad i = 1, \dots, 6, \end{cases} \tag{7}$$

where

$$\begin{cases} \mathcal{B}_1 = \Lambda - \beta S(I + \theta T) + \gamma R - \mu S, \\ \mathcal{B}_2 = \beta S(I + \theta T) - \alpha \dfrac{T}{m+T} E - k_1 E, \\ \mathcal{B}_3 = \omega E - k_2 I, \\ \mathcal{B}_4 = \alpha \dfrac{T}{m+T} E - k_3 I_W, \\ \mathcal{B}_5 = \delta I + \phi I_W - k_4 T, \\ \mathcal{B}_6 = r_1 T + r_2 I_W - k_5 R. \end{cases}$$

Equation (7) can be expressed as,

$$\begin{cases} {}^{C}D_0^{\eta} \mathcal{Q}(\zeta) = \mathcal{Y}(\zeta, \mathcal{Q}(\zeta)), \\ \mathcal{Q}(0) = \mathcal{Q}_0 \geq 0, \end{cases} \tag{8}$$

where $0 \leq \zeta \leq T < \infty,\ 0 < \eta \leq 1$.

Integrating both sides of (8) gives,

$$\mathcal{Q}(\zeta) - \mathcal{Q}(0) = \frac{1}{\Gamma(\eta)} \int_0^{\zeta} (\zeta - \chi)^{\eta-1} \mathcal{Y}(\chi, \mathcal{Q}(\chi)) d\chi. \tag{9}$$

For each class of the proposed model, we can write (9) as,

$$
\begin{cases}
S(\zeta)-S(0)=\dfrac{1}{\Gamma(\eta)}\int_0^\zeta(\zeta-\chi)^{\eta-1}[\mathcal{B}_1(\chi,S(\chi))]d\chi,\\
E(\zeta)-E(0)=\dfrac{1}{\Gamma(\eta)}\int_0^\zeta(\zeta-\chi)^{\eta-1}[\mathcal{B}_2(\chi,E(\chi))]d\chi,\\
I(\zeta)-I(0)=\dfrac{1}{\Gamma(\eta)}\int_0^\zeta(\zeta-\chi)^{\eta-1}[\mathcal{B}_3(\chi,I(\chi))]d\chi,\\
I_W(\zeta)-I_W(0)=\dfrac{1}{\Gamma(\eta)}\int_0^\zeta(\zeta-\chi)^{\eta-1}[\mathcal{B}_4(\chi,I_W(\chi))]d\chi,\\
T(\zeta)-T(0)=\dfrac{1}{\Gamma(\eta)}\int_0^\zeta(\zeta-\chi)^{\eta-1}[\mathcal{B}_5(\chi,T(\chi))]d\chi,\\
R(\zeta)-R(0)=\dfrac{1}{\Gamma(\eta)}\int_0^\zeta(\zeta-\chi)^{\eta-1}[\mathcal{B}_6(\chi,R(\chi))]d\chi.
\end{cases}
\tag{10}
$$

Theorem 4.1. *All kernels $\mathcal{B}_i$ fulfill the Lipschitz condition and they are contraction if,*

$$0\le Q_i<1,\qquad i=1,\dots,6.$$

Proof. Let S and S_1 be two functions, then,

$$\|\mathcal{B}_1(\zeta,S(\zeta))-\mathcal{B}_1(\zeta,S_1(\zeta))\|=\|(\beta(I+\theta T)(S(\zeta)-S_1(\zeta))-\mu(S(\zeta)-S_1(\zeta))\|.$$

Taking $Q_1=|\beta(g_1+\theta g_2)-\mu|,\ \|I(\zeta)\|\le g_1,\ \|T(\zeta)\|\le g_2$, by triangular inequality, we have

$$
\begin{aligned}
\|\mathcal{B}_1(\zeta,S(\zeta))-\mathcal{B}_1(\zeta,S_1(\zeta))\| &\le \|\{\beta(g_1+\theta g_2)-\mu\}\{S(\zeta)-S_1(\zeta)\}\|\\
&\le |\beta(g_1+\theta g_2)-\mu|\|S(\zeta)-S_1(\zeta)\|\\
&\le Q_1\|S(\zeta)-S_1(\zeta)\|.
\end{aligned}
$$

Therefore, the Lipschitz requirement is satisfied for $\mathcal{B}_1$ and if $0\le Q_1<1$, it likewise qualifies as a contraction. Similarly, we have

$$
\begin{cases}
\|\mathcal{B}_2(\zeta,E(\zeta))-\mathcal{B}_2(\zeta,E_1(\zeta))\|\le Q_2\|E(\zeta)-E_1(\zeta)\|,\\
\|\mathcal{B}_3(\zeta,I(\zeta))-\mathcal{B}_3(\zeta,I_1(\zeta))\|\le Q_3\|I(\zeta)-I_1(\zeta)\|,\\
\|\mathcal{B}_4(\zeta,I_W(\zeta))-\mathcal{B}_4(\zeta,I_{W1}(\zeta))\|\le Q_4\|I_W(\zeta)-I_{W1}(\zeta)\|,\\
\|\mathcal{B}_5(\zeta,T(\zeta))-\mathcal{B}_5(\zeta,T_1(\zeta))\|\le Q_5\|T(\zeta)-T_1(\zeta)\|,\\
\|\mathcal{B}_6(\zeta,R(\zeta))-\mathcal{B}_6(\zeta,R_1(\zeta))\|\le Q_6\|R(\zeta)-R_1(\zeta)\|,
\end{cases}
$$

where $Q_2=k_1$, $Q_3=k_2$, $Q_4=k_3$, $Q_5=k_4$, $Q_6=k_5$.

Hence all $\mathcal{B}_i$ meet the Lipschitz requirements, and are contractions if $Q_i\in[0,1)$, $i=2,\dots,6$. □

From (10), the recursive formulas for S, E, I, I_W, T, R are

$$\begin{cases} S_n(\zeta) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_1(\chi, S_{n-1}(\chi)) \Big] d\chi, \\ E_n(\zeta) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_2(\chi, E_{n-1}(\chi)) \Big] d\chi, \\ I_n(\zeta) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_3(\chi, I_{n-1}(\chi)) \Big] d\chi, \\ I_{Wn}(t) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_4(\chi, I_{Wn-1}(\chi)) \Big] d\chi, \\ T_n(\zeta) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_5(\chi, T_{n-1}(\chi)) \Big] d\chi, \\ R_n(\zeta) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_6(\chi, R_{n-1}(\chi)) \Big] d\chi, \end{cases}$$

with initial conditions,

$$\begin{cases} S_0(\zeta) = S(0), & E_0(\zeta) = E(0), & I_0(\zeta) = I(0), \\ I_{W0}(\zeta) = I_W(0), & T_0(\zeta) = T(0), & R_0(\zeta) = R(0). \end{cases}$$

Now, taking the differences ($\Psi_{\tau n}(\zeta)$ where $\tau = 1, \ldots, 6$) between the successive terms yields,

$$\begin{cases} S_n(\zeta) - S_{n-1}(\zeta) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_1(\chi, S_{n-1}(\chi)) - \mathcal{B}_1(\chi, S_{n-2}(\chi)) \Big] d\chi, \\ E_n(\zeta) - E_{n-1}(\zeta) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_2(\chi, E_{n-1}(\chi)) - \mathcal{B}_2(\chi, E_{n-2}(\chi)) \Big] d\chi, \\ I_n(\zeta) - I_{n-1}(\zeta) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_3(\chi, I_{n-1}(\chi)) - \mathcal{B}_3(\chi, I_{n-2}(\chi)) \Big] d\chi, \\ I_{Wn}(\zeta) - I_{Wn-1}(\zeta) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_4(\chi, I_{Wn-1}(\chi)) - \mathcal{B}_4(\chi, I_{Wn-2}(\chi)) \Big] d\chi, \\ T_n(\zeta) - T_{n-1}(\zeta) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_5(\chi, T_{n-1}(\chi)) - \mathcal{B}_5(\chi, T_{n-2}(\chi)) \Big] d\chi, \\ R_n(\zeta) - R_{n-1}(\zeta) = \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_6(\chi, R_{n-1}(\chi)) - \mathcal{B}_6(\chi, R_{n-2}(\chi)) \Big] d\chi. \end{cases} \tag{11}$$

It is clear that,

$$\begin{cases} S_n(\zeta) = \sum_{i=1}^{n} \Psi_{1i}(\zeta), & E_n(\zeta) = \sum_{i=1}^{n} \Psi_{2i}(\zeta), & I_n(\zeta) = \sum_{i=1}^{n} \Psi_{3i}(\zeta), \\ I_{Wn}(\zeta) = \sum_{i=1}^{n} \Psi_{4i}(\zeta), & T_n(\zeta) = \sum_{i=1}^{n} \Psi_{5i}(\zeta), & R_n(t) = \sum_{i=1}^{n} \Psi_{6i}(\zeta). \end{cases}$$

Utilizing the norm concept to (11), we get

$$\|\Psi_{1n}(\zeta)\| = \|S_n(\zeta) - S_{n-1}(\zeta)\| = \left\| \frac{1}{\Gamma(\eta)} \int_0^\zeta (\zeta-\chi)^{\eta-1} \Big[\mathcal{B}_1(\chi, S_{n-1}(\chi)) - \mathcal{B}_1(\chi, S_{n-2}(\chi)) \Big] d\chi \right\|.$$

By using triangular inequality, we have

$$\begin{aligned}\|S_n(\zeta)-S_{n-1}(\zeta)\| &= \left\|\frac{1}{\Gamma(\eta)}\int_0^{\zeta}(\zeta-\chi)^{\eta-1}\Big[\mathcal{B}_1(\chi,S_{n-1}(\chi))-\mathcal{B}_1(\chi,S_{n-2}(\chi))\Big]d\chi\right\| \\ &\leq \frac{1}{\Gamma(\eta)}\int_0^{\zeta}\left\|(\zeta-\chi)^{\eta-1}\Big[\mathcal{B}_1(\chi,S_{n-1}(\chi))-\mathcal{B}_1(\chi,S_{n-2}(\chi))\Big]\right\|d\chi.\end{aligned}$$

Given that the kernel satisfies the Lipschitz condition, then,

$$\|\Psi_{1n}(\zeta)\| \leq \frac{Q_1}{\Gamma(\eta)}\int_0^{\zeta}\|\Psi_{1(n-1)}(\chi)\|d\chi. \tag{12}$$

Similarly, we obtain

$$\|\Psi_{in}(\zeta)\| \leq \frac{Q_i}{\Gamma(\eta)}\int_0^{\zeta}\|\Psi_{i(n-1)}(\chi)\|d\chi, \qquad \forall\ i=2,\ldots,6. \tag{13}$$

Theorem 4.2. *For a finite time ζ_0, if*

$$\frac{\zeta_0}{\Gamma(\eta)}Q_i<1, \qquad \forall\ i=1,\ldots,6,$$

then, Model (2) solution exists.

Proof. By using (12) and (13), and recursive principle [25], we obtain

$$\begin{cases}\|\Psi_{1n}(\zeta)\| \leq \|S(0)\|\left[\frac{Q_1}{\Gamma(\eta)}\right]^n,\\ \|\Psi_{2n}(\zeta)\| \leq \|E(0)\|\left[\frac{Q_2}{\Gamma(\eta)}\right]^n,\\ \|\Psi_{3n}(\zeta)\| \leq \|I(0)\|\left[\frac{Q_3}{\Gamma(\eta)}\right]^n,\\ \|\Psi_{4n}(\zeta)\| \leq \|I_W(0)\|\left[\frac{Q_4}{\Gamma(\eta)}\right]^n,\\ \|\Psi_{5n}(\zeta)\| \leq \|T(0)\|\left[\frac{Q_5}{\Gamma(\eta)}\right]^n,\\ \|\Psi_{6n}(\zeta)\| \leq \|R(0)\|\left[\frac{Q_6}{\Gamma(\eta)}\right]^n.\end{cases} \tag{14}$$

Hence, the Model (2) has at least one solution. Now, we will demonstrate that the aforementioned functions yield the solution for Model (2). Let,

$$\begin{cases}S(\zeta)-S(0)=S_n(\zeta)-B_{1n}(\zeta),\\ E(\zeta)-E(0)=E_n(\zeta)-B_{2n}(\zeta),\\ I(\zeta)-I(0)=I_n(\zeta)-B_{3n}(\zeta),\\ I_W(\zeta)-I_W(0)=I_{Wn}(\zeta)-B_{4n}(\zeta),\\ T(\zeta)-T(0)=T_n(\zeta)-B_{5n}(\zeta),\\ R(\zeta)-R(0)=R_n(\zeta)-B_{6n}(\zeta),\end{cases}$$

thus, by the triangular inequality, we have

$$\begin{aligned}\|\Pi_{1n}(\zeta)\| &= \left\|\frac{1}{\Gamma(\eta)}\int_0^{\zeta}(\zeta-\chi)^{\eta-1}\Big[\mathcal{B}_1(\chi,S(\chi))-\mathcal{B}_1(\chi,S_{n-1}(\chi))\Big]d\chi\right\| \\ &\le \frac{1}{\Gamma(\eta)}\int_0^{\zeta}(\zeta-\chi)^{\eta-1}\Big[\|\mathcal{B}_1(\chi,S(\chi))-\mathcal{B}_1(\chi,S_{n-1}(\chi))\|\Big]d\chi \\ &\le \frac{\zeta}{\Gamma(\eta)}Q_1\|S-S_{n-1}\|.\end{aligned}$$

Using the recursive principle, gives,

$$\|\Pi_{1n}(\zeta)\| \le \left(\frac{\zeta}{\Gamma(\eta)}Q_1\right)^{n+1} a.$$

At ζ_0, we have

$$\|\Pi_{1n}(\zeta_0)\| \le \left(\frac{\zeta_0}{\Gamma(\eta)}Q_1\right)^{n+1} a.$$

Thus,

$$\|\Pi_{1n}(\zeta)\| \to 0, \quad \text{as} \quad n\to\infty, \quad \text{if} \quad \frac{\zeta_0}{\Gamma(\eta)}Q_1 < 1.$$

Similar,

$$\|\Pi_{in}(\zeta)\| \to 0, \quad \text{as} \quad n\to\infty, \quad i=2,\dots,6.$$

Hence proved. □

Theorem 4.3. *If* $\left[1-\frac{\zeta}{\Gamma(\eta)}Q_i\right] > 0, \quad \forall\, i=1,\dots,6,$ *then, the aforementioned Model* (2) *has a unique solution.*

Proof. Let S and S_1 be two functions, and to show the uniqueness of the solution of Model (2), then, we have

$$S(\zeta)-S_1(\zeta) = \frac{1}{\Gamma(\eta)}\int_0^{\zeta}(\zeta-\chi)^{\eta-1}\Big[\mathcal{B}_1(\chi,S(\chi))-\mathcal{B}_1(\chi,S_1(\chi))\Big]d\chi, \tag{15}$$

it is obvious that,

$$\|S(\zeta)-S_1(\zeta)\| \ge 0.$$

Utilizing the norm concept (15), we get

$$\|S(\zeta)-S_1(\zeta)\| \le \frac{1}{\Gamma(\eta)}\int_0^{\zeta}(\zeta-\chi)^{\eta-1}\Big[\|\mathcal{B}_1(\chi,S(\chi))-\mathcal{B}_1(\chi,S_1(\chi))\|\Big]d\chi.$$

By Lipschitz condition, we obtain

$$\begin{aligned}\|S(\zeta)-S_1(\zeta)\| &\le \frac{\zeta}{\Gamma(\eta)}Q_1\Big[\|S(\zeta)-S_1(\zeta)\|\Big] \\ \left(1-\frac{\zeta}{\Gamma(\eta)}Q_1\right)&\Big[\|S(\zeta)-S_1(\zeta)\|\Big] \le 0.\end{aligned}$$

Since,

$$\frac{\zeta}{\Gamma(\eta)}Q_1 < 1,$$

we have

$$\|S(\zeta) - S_1(\zeta)\| = 0,$$

which gives,

$$S(\zeta) = S_1(\zeta).$$

Similarly,

$$E(\zeta) = E_1(\zeta), \quad I(\zeta) = I_1(\zeta), \quad I_W(\zeta) = I_{W1}(\zeta), \quad T(\zeta) = T_1(\zeta), \quad R(\zeta) = R_1(\zeta).$$

Hence, the Model (2) has an unique solution. □

5 Fundamental Characteristics

The foundational properties of the solutions of the given Model (2), such as positivity, boundedness, and invariant region, are presented in this section.

5.1 Invariant region

Let $\Omega = \{(S, E, I, I_W, T, R) \in R_+^6\}$, $N(\zeta) = S(\zeta) + E(\zeta) + I(\zeta) + I_W(\zeta) + T(\zeta) + R(\zeta)$ feasible region, and all functions of Model (2) are continuous on R_+^6. Adding all classes of population, then, the net population becomes,

$${}^C D_0^\eta N(\zeta) = {}^C D_0^\eta S + {}^C D_0^\eta E + {}^C D_0^\eta I + {}^C D_0^\eta I_W + {}^C D_0^\eta T + {}^C D_0^\eta R,$$

which gives,

$${}^C D_0^\eta N(\zeta) = \Lambda - \mu N(\zeta). \tag{16}$$

Applying the Laplace transformation to (16), we have

$$L\left\{{}^C D_0^\eta N(\zeta)\right\} = \Lambda L\{1\} - \mu L\{N(\zeta)\}.$$
$$s^\eta L\{N(\zeta)\} - s^{\eta-1}N(0) = \frac{\Lambda}{s} - \mu L\{N(\zeta)\}.$$
$$(s^\eta + \mu)L\{N(\zeta)\} = s^{\eta-1}N(0) + \frac{\Lambda}{s}.$$

Hence,

$$L\{N(\zeta)\} = \frac{s^{-1}}{s^\eta + \mu}\Lambda + \frac{s^{\eta-1}}{s^\eta + \mu}N(0).$$

Using Definition 3.2 and inverse Laplace transformation on both sides, we have

$$N(\zeta) = \Lambda\zeta^\eta E_{\eta,\eta+1}(-\mu\zeta^\eta) + N(0)E_{\eta,1}(-\mu\zeta^\eta),$$
$$N(\zeta) = \mu\frac{\Lambda}{\mu}\zeta^\eta E_{\eta,\eta+1}(-\mu\zeta^\eta) + N(0)E_{\eta,1}(-\mu\zeta^\eta).$$

Let $M = \max\left\{N(0), \dfrac{\Lambda}{\mu}\right\}$, then, we have

$$N(\zeta) \leq M\left\{\mu\zeta^{\eta}E_{\eta,\eta+1}(-\mu\zeta^{\eta}) + E_{\eta,1}(-\mu\zeta^{\eta})\right\}.$$

By using $E_{\eta,q}(\zeta) = \zeta E_{\eta,\eta+q}(\zeta) + \dfrac{1}{\Gamma q}$, we get

$$\begin{aligned} N(\zeta) &\leq M\{-\mu\zeta^{\eta}E_{\eta,\eta+1}(-\mu\zeta^{\eta}) + \frac{1}{\Gamma(1)} + \mu\zeta^{\eta}E_{\eta,\eta+1}(-\mu\zeta^{\eta})\} \\ N(\zeta) &\leq M. \end{aligned}$$

Hence, $N(\zeta)$ is bounded uniformly and given Model (2) all solutions are uniformly bounded in Ω. Thus, the region Ω is positively invariant.

5.2 Positivity and boundedness

In order to demonstrate that every solution of the suggested Model (2) is positive, we assess the Model (2) first equation as,

$$\begin{aligned} {}^{C}D_0^{\eta}S(\zeta) &= \Lambda - \beta S(I+\theta T) + \gamma R - \mu S \\ &\geq -(\beta(I+\theta T)+\mu)S \\ &\geq -\mu S. \end{aligned} \tag{17}$$

Using Lemma 9 from [12] and $E_{\eta,1}(\zeta) > 0$, for any $\eta \in (0, 1]$, (17) becomes

$$S(\zeta) \geq S(0)E_{\eta,1}(-\mu\zeta^{\eta}) \Rightarrow S(\zeta) \geq 0.$$

Similarly, other equations of the Model (2) are positive. Therefore,

$$\Omega = \{(S, E, I, I_W, T, R) \in R_+^6 \mid (S + E + I + I_W + T + R) \geq 0\}.$$

Therefore, the Model (2) solutions are both positive and bounded to the feasible region Ω.

5.3 Fundamental properties

In the section, we provides DFEP, basic reproduction number, and sensitivity analysis.

5.3.1 DFEP ($\mathbf{E_0}$)

To study the DFEP of the Model (2), let,

$${}^{C}D_0^{\eta}S = {}^{C}D_0^{\eta}E = {}^{C}D_0^{\eta}I = {}^{C}D_0^{\eta}I_W = {}^{C}D_0^{\eta}T = {}^{C}D_0^{\eta}R = 0.$$

Now, considering $\mathbf{E_0}$ is DFEP such that,

$$\mathbf{E_0}(S, E, I, I_W, T, R) = (S_0, E_0, I_0, I_{W0}, T_0, R_0).$$

By simplification and calculation, we obtain

$$\mathbf{E_0} = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0\right).$$

5.3.2 Basic reproduction number ($\mathbf{R_0}$)

The Basic reproduction number $\mathbf{R_0}$ of the model described in (2) is ascertained using the next-generation matrix method [39].

To calculate $\mathbf{R_0}$, we analyze the simplified system $X = (E, I, I_W, T)^T$ and deduce the necessary matrices $\mathcal{F}$ and $\mathcal{V}$. The Jacobian matrices evaluated at $\mathbf{E_0}$ are as follows,

$$\mathcal{F}(\mathbf{E_0}) = \begin{bmatrix} 0 & \dfrac{\beta\Lambda}{\mu} & 0 & \dfrac{\beta\Lambda\theta}{\mu} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{V}(\mathbf{E_0}) = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ -\omega & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -\delta & -\phi & k_4 \end{bmatrix},$$

and

$$\mathcal{F}\mathcal{V}^{-1} = \begin{bmatrix} \dfrac{\beta\Lambda\omega(k_4+\theta\phi)}{k_1k_2k_4\mu} & \dfrac{\beta\Lambda(k_4+\delta\theta)}{k_2k_4\mu} & \dfrac{\beta\Lambda\theta\phi}{k_3k_4\mu} & \dfrac{\beta\Lambda\theta}{k_4\mu} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The $\mathbf{R_0}$ is the spectral radius ρ of $\mathcal{F}\mathcal{V}^{-1}$. After simplification of $\mathcal{F}\mathcal{V}^{-1}$, we have

$$\mathbf{R_0} = \rho(\mathcal{F}\mathcal{V}^{-1}) = \frac{\beta\Lambda\omega(k_4+\theta\phi)}{k_1k_2k_4\mu}.$$

5.4 Sensitivity analysis

We want to see how $\mathbf{R_0}$ is sensitive or responsive to small changes in the different factors or parameters that fit into the disease model. To do this, we compute the sensitivity index for each parameter. The sensitivity index tells us how much $\mathbf{R_0}$ changes will shift if we tweak that specific parameter a little bit, while keeping all the other parameters the same. We employ an approach similar to that utilized by Nawaz et al. [28] to figure out the normalized forward sensitivity index of $\mathbf{R_0}$ for a specific parameter. The sensitivity index $\mathbf{R_0}$ concerning the model parameters is derived by,

$$\Upsilon_{\omega}^{R_0} = \frac{\beta\Lambda(\phi\theta+k_4)(1-\omega)}{k_1A} > 0, \qquad \Upsilon_{\delta}^{R_0} = -\frac{\beta\Lambda\omega(k_4+\theta\phi)}{k_2A} < 0, \qquad \Upsilon_{r_1}^{R_0} = -\frac{\beta\Lambda\theta\omega\phi}{k_4A} < 0,$$

$$\Upsilon_{\Lambda}^{R_0} = \frac{\beta\omega(k_4+\theta\phi)}{A} > 0, \qquad \Upsilon_{\beta}^{R_0} = \frac{\Lambda\omega(k_4+\theta\phi)}{A} > 0, \qquad \Upsilon_{\theta}^{R_0} = \frac{\beta\Lambda\omega\phi}{A} > 0, \qquad \Upsilon_{\phi}^{R_0} = \frac{\beta\Lambda\theta\delta}{A} > 0,$$

$$\Upsilon_{\mu}^{R_0} = -\left[\frac{\beta\Lambda\omega\left[k_4(k_2\mu+k_1(\mu+k_2))+\phi\theta(k_4\mu(k_1+k_2)+k_1k_2(\mu+k_4))\right]}{A^2}\right] < 0,$$

where $A = k_1k_2k_4\mu$.

It is readily apparent that the parameters $\Lambda,\ \theta,\ \phi,\ \omega$, and β are associated with positive indices. This implies that the value of the basic threshold ratio will drop when these parameters rise. On the other hand, indices of the parameters $\mu,\ \delta,\ \mu$ and r_1 show negative indications. This implies that a change in these values will raise $\mathbf{R_0}$. Figure 2 displays the partial rank correlation coefficient results for the significance of parameters included in $\mathbf{R_0}$.

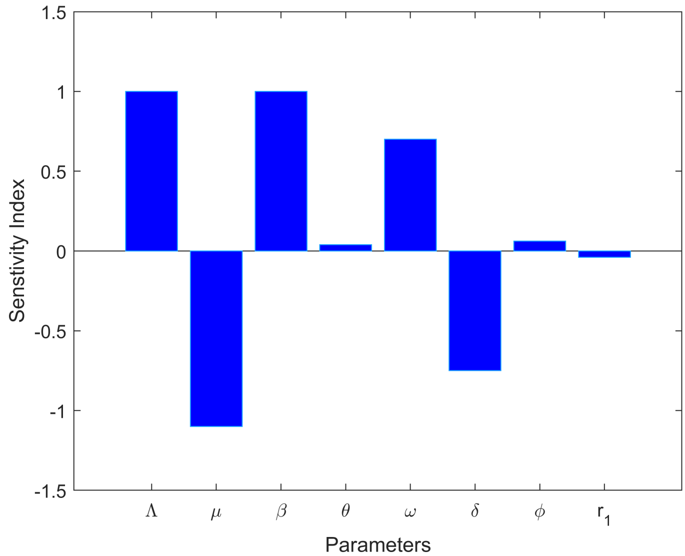

Figure 2: Sensitivity indices.

Figure 3 displays the ratio influence of infected individuals seeking medication, with media influenced α is positive. Attributed to the media, the graph reveals a notable rise in knowledge among seasonal farm workers and infected immigrants. This implies that media awareness can help to control TB. Which means that the disease can be controlled with media awareness.

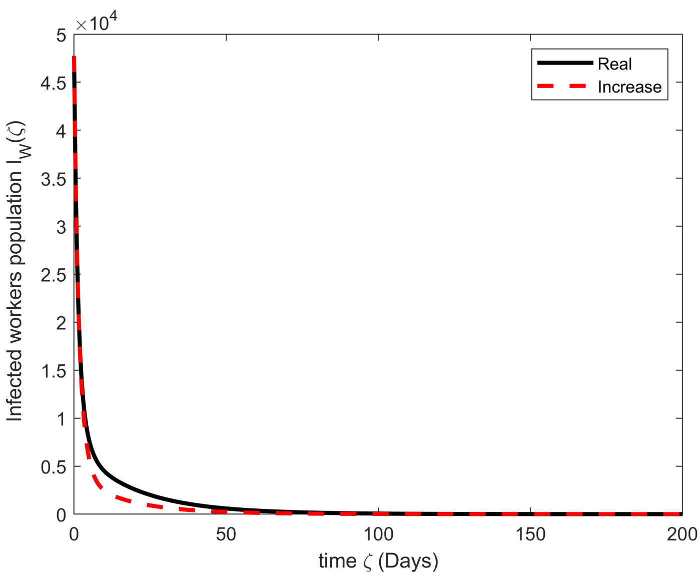

Figure 3: Analysis of media impact parameter α on infected workers $I_W(\eta)$.

6 HU Stability

This section addresses the HU stability of Model (2) and establishes the necessary inequalities. We define the required inequalities,

$$\left|{}^{C}D_0^{\eta}\check{\mathcal{Q}}(\zeta) - \mathcal{Y}(\zeta, \check{\mathcal{Q}}(\zeta))\right| \leq \epsilon, \qquad \zeta \in J. \tag{18}$$

where $\epsilon = \max(\epsilon_i)^T$, $i = 1, \ldots, 6$.

Definition 6.1. *TB Model (2) is HU stable, if there are constants $\mathcal{G}_i > 0$, $i \in \mathcal{N}_1^6$, such that for every $\epsilon > 0$ and a solution $\check{\mathcal{Q}}$ satisfying (18), there is a unique solution $\mathcal{Q}$ with,*

$$\left|\check{\mathcal{Q}}(\zeta) - \mathcal{Q}(\zeta)\right| \leq \mathcal{G}_i\epsilon, \qquad i = 1, \ldots, 6. \tag{19}$$

Remark 6.1. *Consider $\check{\mathcal{Q}}$ is the solution of inequality (18), and if there are h_i, $i = 1, \ldots, 6$, such that $|h_i(\zeta)| < \epsilon_i$, we have*

$${}^{C}D_0^{\eta}\check{\mathcal{Q}}(\zeta) = \mathcal{Y}(\zeta, \check{\mathcal{Q}}(\zeta)) + h_i(\zeta), \qquad i = 1, \ldots, 6. \tag{20}$$

Theorem 6.1. *Let,*

$$\frac{\alpha_i\zeta}{\Gamma(\eta)} < 1, \qquad i = 1, \ldots, 6,$$

holds. Then, the proposed TB Model (2) is Hyers-Ulam stable.

Proof. From Remark 6.1, we have

$${}^{C}D_0^{\eta}\check{\mathcal{Q}}(\zeta) = \mathcal{Y}(\zeta, \check{\mathcal{Q}}(\zeta)) + h_i(\zeta),$$

which gives,

$$\check{\mathcal{Q}}(\zeta) = \check{\mathcal{Q}}(0) + \frac{1}{\Gamma(\eta)}\int_0^{\zeta}(\zeta - y)^{\eta-1}\mathcal{Y}(y, \check{\mathcal{Q}}(y))dy + \frac{1}{\Gamma(\eta)}\int_0^{\zeta}(\zeta - y)^{\eta-1}h_i(y)dy.$$

Let $\mathcal{Q}_(\zeta)$ be a unique solution of the proposed model, then,

$$\mathcal{Q}(z) = \mathcal{Q}(0) + \frac{1}{\Gamma(\eta)}\int_0^{z}(z - y)^{\eta-1}\mathcal{Y}(y, \mathcal{Q}(y))dy.$$

Hence,

$$\begin{aligned}
\left\|\check{\mathcal{Q}} - \mathcal{Q}\right\| &\leq \frac{1}{\Gamma(\eta)}\int_0^{\zeta}(\zeta - y)^{\eta-1}\left\|\mathcal{Y}(y, \check{\mathcal{Q}}(y)) - \mathcal{Y}(y, \mathcal{Q}(y))\right\|dy \\
&\quad + \frac{1}{\Gamma(\eta)}\int_0^{\zeta}(\zeta - y)^{\eta-1}\left\|h_i(y)\right\|dy \\
&\leq \frac{\zeta}{\Gamma(\eta)}\alpha_1\left\|\check{\mathcal{Q}} - \mathcal{Q}\right\| + \frac{\zeta}{\Gamma(\eta)}\epsilon_1 \\
\|\check{\mathcal{Q}} - \mathcal{Q}\| &\leq \frac{\left[\frac{\zeta}{\Gamma(\eta)}\right]}{\left[1 - \frac{\alpha_i\zeta}{\Gamma(\eta)}\right]}\epsilon.
\end{aligned}$$

Then,

$$\|\check{\mathcal{Q}} - \mathcal{Q}\| \leq \mathcal{G}_i \epsilon,$$

where $\mathcal{G}_i = \dfrac{\left[\dfrac{\zeta}{\Gamma(\eta)}\right]}{\left[1 - \dfrac{\alpha_i \zeta}{\Gamma(\eta)}\right]}$, $i = 1, \ldots, 6$. Hence, Model (2) is HU stable. □

7 Numerical Simulation

This section addresses a numerical simulation utilizing the Caputo fractional operator to demonstrate the dynamic behavior of the proposed TB model. The mathematical scheme of the MEM is utilized to obtain an approximate solution of the problem under consideration. The proposed model can be written as,

$$\begin{cases} {}^C D_0^\eta S(\zeta) = \mathcal{B}_1(S, E, I, I_W, T, R) = \Lambda - \beta S(I + \theta T) + \gamma R - \mu S, \\ {}^C D_0^\eta E(\zeta) = \mathcal{B}_2(S, E, I, I_W, T, R) = \beta S(I + \theta T) - \alpha \dfrac{T}{m + T} E - k_1 E, \\ {}^C D_0^\eta I(\zeta) = \mathcal{B}_3(S, E, I, I_W, T, R) = \omega E - k_2 I, \\ {}^C D_0^\eta I_W(\zeta) = \mathcal{B}_4(S, E, I, I_W, T, R) = \alpha \dfrac{T}{m + T} E - k_3 I_W, \\ {}^C D_0^\eta T(\zeta) = \mathcal{B}_5(S, E, I, I_W, T, R) = \delta I + \phi I_W - k_4 T, \\ {}^C D_0^\eta R(\zeta) = \mathcal{B}_6(S, E, I, I_W, T, R) = r_1 T + r_2 I_W - k_5 R. \end{cases} \tag{21}$$

Let $[0, T]$ be the interval of the solution for (21). We divide $[0, T]$ into m sub-intervals $\zeta \in [\zeta_j, \zeta_{j+1}]$ with equal width $h = \dfrac{T}{m}$ by using the nodes $\zeta_j = jh$ for $j = 0, 1, \ldots, m$. Consider $S(\zeta)$, $E(\zeta)$, $I(\zeta)$, $I_W(\zeta)$, $T(\zeta)$, $R(\zeta)$, ${}^C D_0^\eta S(\zeta)$, ${}^C D_0^\eta E(\zeta)$, ${}^C D_0^\eta I(\zeta)$, ${}^C D_0^\eta I_W(\zeta)$, ${}^C D_0^\eta T(\zeta)$, ${}^C D_0^\eta R(\zeta)$, and upto higher orders are continuous on $[0, T]$. We use MEM to expand $S(\zeta)$, $E(\zeta)$, $I(\zeta)$, $I_W(\zeta)$, $T(\zeta)$ and $R(\zeta)$ for $\zeta = \zeta_0 = 0$. The expression for ζ_1 is

$$\begin{cases} S(\zeta_1) = S(\zeta_0) + \mathcal{B}_1(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0)) \dfrac{h^\eta}{\Gamma(\eta + 1)} + {}^C D_0^{2\eta} S(\zeta) \dfrac{h^{2\eta}}{\Gamma(2\eta + 1)}, \\ E(\zeta_1) = E(\zeta_0) + \mathcal{B}_2(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0)) \dfrac{h^\eta}{\Gamma(\eta + 1)} + {}^C D_0^{2\eta} E(\zeta) \dfrac{h^{2\eta}}{\Gamma(2\eta + 1)}, \\ I(\zeta_1) = I(\zeta_0) + \mathcal{B}_3(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0)) \dfrac{h^\eta}{\Gamma(\eta + 1)} + {}^C D_0^{2\eta} I(\zeta) \dfrac{h^{2\eta}}{\Gamma(2\eta + 1)}, \\ I_W(\zeta_1) = I_W(\zeta_0) + \mathcal{B}_4(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0)) \dfrac{h^\eta}{\Gamma(\eta + 1)} + {}^C D_0^{2\eta} I_W(\zeta) \dfrac{h^{2\eta}}{\Gamma(2\eta + 1)}, \\ T(\zeta_1) = T(\zeta_0) + \mathcal{B}_5(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0)) \dfrac{h^\eta}{\Gamma(\eta + 1)} + {}^C D_0^{2\eta} T(\zeta) \dfrac{h^{2\eta}}{\Gamma(2\eta + 1)}, \\ R(\zeta_1) = R(\zeta_0) + \mathcal{B}_6(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0)) \dfrac{h^\eta}{\Gamma(\eta + 1)} + {}^C D_0^{2\eta} R(\zeta) \dfrac{h^{2\eta}}{\Gamma(2\eta + 1)}. \end{cases}$$

Given the diminutive step size h in this method, we disregard the second-order expressions that include $h^{2\eta}$. Subsequently, we possess

$$\begin{cases} S(\zeta_1) = S(\zeta_0) + \mathcal{B}_1(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ E(\zeta_1) = E(\zeta_0) + \mathcal{B}_2(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ I(\zeta_1) = I(\zeta_0) + \mathcal{B}_3(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ I_W(\zeta_1) = I_W(\zeta_0) + \mathcal{B}_4(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ T(\zeta_1) = T(\zeta_0) + \mathcal{B}_5(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ R(\zeta_1) = R(\zeta_0) + \mathcal{B}_6(S(\zeta_0), E(\zeta_0), I(\zeta_0), I_W(\zeta_0), T(\zeta_0), R(\zeta_0))\dfrac{h^\eta}{\Gamma(\eta+1)}. \end{cases}$$

Furthermore, the subsequent terms are

$$\begin{cases} S(\zeta_2) = S(\zeta_1) + \mathcal{B}_1(S(\zeta_1), E(\zeta_1), I(\zeta_1), I_W(\zeta_1), T(\zeta_1), R(\zeta_1))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ E(\zeta_2) = E(\zeta_1) + \mathcal{B}_2(S(\zeta_1), E(\zeta_1), I(\zeta_1), I_W(\zeta_1), T(\zeta_1), R(\zeta_1))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ I(\zeta_2) = I(\zeta_1) + \mathcal{B}_3(S(\zeta_1), E(\zeta_1), I(\zeta_1), I_W(\zeta_1), T(\zeta_1), R(\zeta_1))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ I_W(\zeta_2) = I_W(\zeta_1) + \mathcal{B}_4(S(\zeta_1), E(\zeta_1), I(\zeta_1), I_W(\zeta_1), T(\zeta_1), R(\zeta_1))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ T(\zeta_2) = T(\zeta_1) + \mathcal{B}_5(S(\zeta_1), E(\zeta_1), I(\zeta_1), I_W(\zeta_1), T(\zeta_1), R(\zeta_1))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ R(\zeta_2) = R(\zeta_1) + \mathcal{B}_6(S(\zeta_1), E(\zeta_1), I(\zeta_1), I_W(\zeta_1), T(\zeta_1), R(\zeta_1))\dfrac{h^\eta}{\Gamma(\eta+1)}. \end{cases}$$

Similarly, the iteration at $\zeta_{j+1} = \zeta_j + h$ (where $j = 0, 1, \ldots, m$) is as follows,

$$\begin{cases} S(\zeta_{j+1}) = S(\zeta_j) + \mathcal{B}_1(S(\zeta_j), E(\zeta_j), I(\zeta_j), I_W(\zeta_j), T(\zeta_j), R(\zeta_j))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ E(\zeta_{j+1}) = E(\zeta_j) + \mathcal{B}_2(S(\zeta_j), E(\zeta_j), I(\zeta_j), I_W(\zeta_j), T(\zeta_j), R(\zeta_j))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ I(\zeta_{j+1}) = I(\zeta_j) + \mathcal{B}_3(S(\zeta_j), E(\zeta_j), I(\zeta_j), I_W(\zeta_j), T(\zeta_j), R(\zeta_j))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ I_W(\zeta_{j+1}) = I_W(\zeta_j) + \mathcal{B}_4(S(\zeta_j), E(\zeta_j), I(\zeta_j), I_W(\zeta_j), T(\zeta_j), R(\zeta_j))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ T(\zeta_{j+1}) = T(\zeta_j) + \mathcal{B}_5(S(\zeta_j), E(\zeta_j), I(\zeta_j), I_W(\zeta_j), T(\zeta_j), R(\zeta_j))\dfrac{h^\eta}{\Gamma(\eta+1)}, \\ R(\zeta_{j+1}) = R(\zeta_j) + \mathcal{B}_6(S(\zeta_j), E(\zeta_j), I(\zeta_j), I_W(\zeta_j), T(\zeta_j), R(\zeta_j))\dfrac{h^\eta}{\Gamma(\eta+1)}. \end{cases}$$

After further simplification, we obtain the general approximate solution of proposed model as,

$$\begin{aligned}
S(\zeta_j) &= S(0) + \frac{h^\eta}{\Gamma(\eta+2)}((j-1)^{\eta+1} - (j-\eta-1)j^\eta)\mathcal{B}_1(\zeta_0, S(\zeta_0)) + \frac{h^\eta}{\Gamma(\eta+2)}\sum_{i=1}^{j-1}((j-i+1)^{\eta+1} \\
&\quad - 2(j-1)^{\eta+1} + (j-i-1)^{\eta+1})\mathcal{B}_1(\zeta_i, S(\zeta_i)) + \frac{h^\eta}{\Gamma(\eta+2)}\mathcal{B}_1(\zeta_j, S(\zeta_{j-1})) \\
&\quad + \frac{h^\eta}{\Gamma(\eta+1)}\mathcal{B}_1(\zeta_{j-1}, S(\zeta_{j-1})), \\
E(\zeta_j) &= E(0) + \frac{h^\eta}{\Gamma(\eta+2)}((j-1)^{\eta+1} - (j-\eta-1)j^\eta)\mathcal{B}_2(\zeta_0, E(\zeta_0)) + \frac{h^\eta}{\Gamma(\eta+2)}\sum_{i=1}^{j-1}((j-i+1)^{\eta+1} \\
&\quad - 2(j-1)^{\eta+1} + (j-i-1)^{\eta+1})\mathcal{B}_2(\zeta_i, E(\zeta_i)) + \frac{h^\eta}{\Gamma(\eta+2)}\mathcal{B}_2(\zeta_j, E(\zeta_{j-1})) \\
&\quad + \frac{h^\eta}{\Gamma(\eta+1)}\mathcal{B}_2(\zeta_{j-1}, E(\zeta_{j-1})), \\
I(\zeta_j) &= I(0) + \frac{h^\eta}{\Gamma(\eta+2)}((j-1)^{\eta+1} - (j-\eta-1)j^\eta)\mathcal{B}_3(\zeta_0, I(\zeta_0)) + \frac{h^\eta}{\Gamma(\eta+2)}\sum_{i=1}^{j-1}((j-i+1)^{\eta+1} \\
&\quad - 2(j-1)^{\eta+1} + (j-i-1)^{\eta+1})\mathcal{B}_3(\zeta_i, I(\zeta_i)) + \frac{h^\eta}{\Gamma(\eta+2)}\mathcal{B}_3(\zeta_j, I(z_{j-1})) \\
&\quad + \frac{h^\eta}{\Gamma(\eta+1)}\mathcal{B}_3(\zeta_{j-1}, I(\zeta_{j-1})), \\
I_W(\zeta_j) &= I_W(0) + \frac{h^\eta}{\Gamma(\eta+2)}((j-1)^{\eta+1} - (j-\eta-1)j^\eta)\mathcal{B}_4(\zeta_0, I_W(\zeta_0)) + \frac{h^\eta}{\Gamma(\eta+2)}\sum_{i=1}^{j-1}((j-i+1)^{\eta+1} \\
&\quad - 2(j-1)^{\eta+1} + (j-i-1)^{j-1})\mathcal{B}_4(\zeta_i, I_W(\zeta_i)) + \frac{h^\eta}{\Gamma(\eta+2)}\mathcal{B}_4(\zeta_j, I_W(\zeta_{j-1})) \\
&\quad + \frac{h^\eta}{\Gamma(\eta+1)}\mathcal{B}_4(\zeta_{j-1}, I_W(\zeta_{j-1})), \\
T(\zeta_j) &= T(0) + \frac{h^\eta}{\Gamma(\eta+2)}((j-1)^{\eta+1} - (j-\eta-1)j^\eta)\mathcal{B}_5(\zeta_0, T(\zeta_0)) + \frac{h^\eta}{\Gamma(\eta+2)}\sum_{i=1}^{j-1}((j-i+1)^{\eta+1} \\
&\quad - 2(j-1)^{\eta+1} + (j-i-1)^{\eta+1})\mathcal{B}_5(\zeta_i, T(\zeta_i)) + \frac{h^\eta}{\Gamma(\eta+2)}\mathcal{B}_5(\zeta_j, T(\zeta_{j-1})) \\
&\quad + \frac{h^\eta}{\Gamma(\eta+1)}\mathcal{B}_5(\zeta_{j-1}, T(\zeta_{j-1})), \\
R(\zeta_j) &= R(0) + \frac{h^\eta}{\Gamma(\eta+2)}((j-1)^{\eta+1} - (j-\eta-1)j^\eta)\mathcal{B}_6(\zeta_0, R(\zeta_0)) + \frac{h^\eta}{\Gamma(\eta+2)}\sum_{i=1}^{j-1}((j-i+1)^{\eta+1} \\
&\quad - 2(j-1)^{\eta+1} + (j-i-1)^{\eta+1})\mathcal{B}_6(\zeta_i, R(\zeta_i)) + \frac{h^\eta}{\Gamma(\eta+2)}\mathcal{B}_6(\zeta_j, R(\zeta_{j-1})) \\
&\quad + \frac{h^\eta}{\Gamma(\eta+1)}\mathcal{B}_6(\zeta_{j-1}, R(\zeta_{j-1})),
\end{aligned}$$

The simulation parameters are taken from [38] shown in Table 1. Initial conditions of all compartments are as follows, $S(0) = 23127003$, $E(0) = 630380$, $I(0) = 60706$, $I_W(0) = 261929$, $T(0) = 47742$, $R(0) = 672240$.

Table 1: Description of parameters for the Model (2).

Parameter	Description	Value
Λ	Recruitment rate by migrants and birth	385600
μ	Natural mortality rate	0.01382
γ	Rate of recovered individuals regain susceptibility	0.05
β	Rate of transmission	1.937×10^{-8}
θ	Reduction coefficient of I enter into T	0.0471
ω	Rate of exposed population becomes infected	0.0518
δ	Transmission ratio from I to T	0.6541
ϕ	Transmission ratio from I_W to T	0.0222
d_1	Disease-induced mortality rate of compartment I	0.1128
d_2	Disease-induced mortality rate of compartment T	0.0291
r_1	Rate of recovery form compartment T	0.6062
r_2	Rate of recovery form compartment I_W	0.8678
α	Impact rate influenced by media	0.0164
m	Half saturation constant	36704

We simulate all compartments for numerous fractional orders and an integer order $\eta = 1$. In Figures 4(a)– 4(f), we present the numerical simulations for each compartment of the model corresponding to different values of the fractional derivative order, $\eta = 0.75, 0.85, 0.95, 1$, plotted against time ζ (in days). The convergence rate of each compartment is notably influenced by the fractional order η. Although the quantitative results vary across different values of η, the overall qualitative behavior remains consistent across all cases. Figure 4(a) illustrates a gradual increase in the susceptible population over time, eventually stabilizing after approximately 200 days. Figure 4(b) shows a decreasing trend in the exposed population, with the rate of decline depending on the fractional order. Similarly, Figure 4(c) demonstrates a reduction in the infected population over time. Figures 4(d) and 4(e) further reveal a steady decline in both infected workers and those under treatment, suggesting a gradual reduction in disease burden and improvement in overall community health.

Both graphs show convergence and stability. The proportion of cured individuals increases and after a few days decreases with the adoption of appropriate treatment, as shown in Figure 4(f). All graphs show convergence and stability of the proposed model. The population grows or declines more rapidly at lower fractional orders, but this pattern reverses as the fractional order increases in a specific class. From the graphical observation, the health improvement can be expected in this population over time. Moreover, the graphical results show that the two key parameters, α and m, have a distinct influence on the rate of the infected workers population. To attain the objective of total TB eradication, the parameter α can affect the rate of TB cases. But m can only slightly decline the rate of TB cases. The incidence and eventual elimination of TB can be regulated by the transmission rate, treatment, and recovery. One important way to improve TB control among seasonal farm workers and migrants is to encourage them to seek preventive treatment.

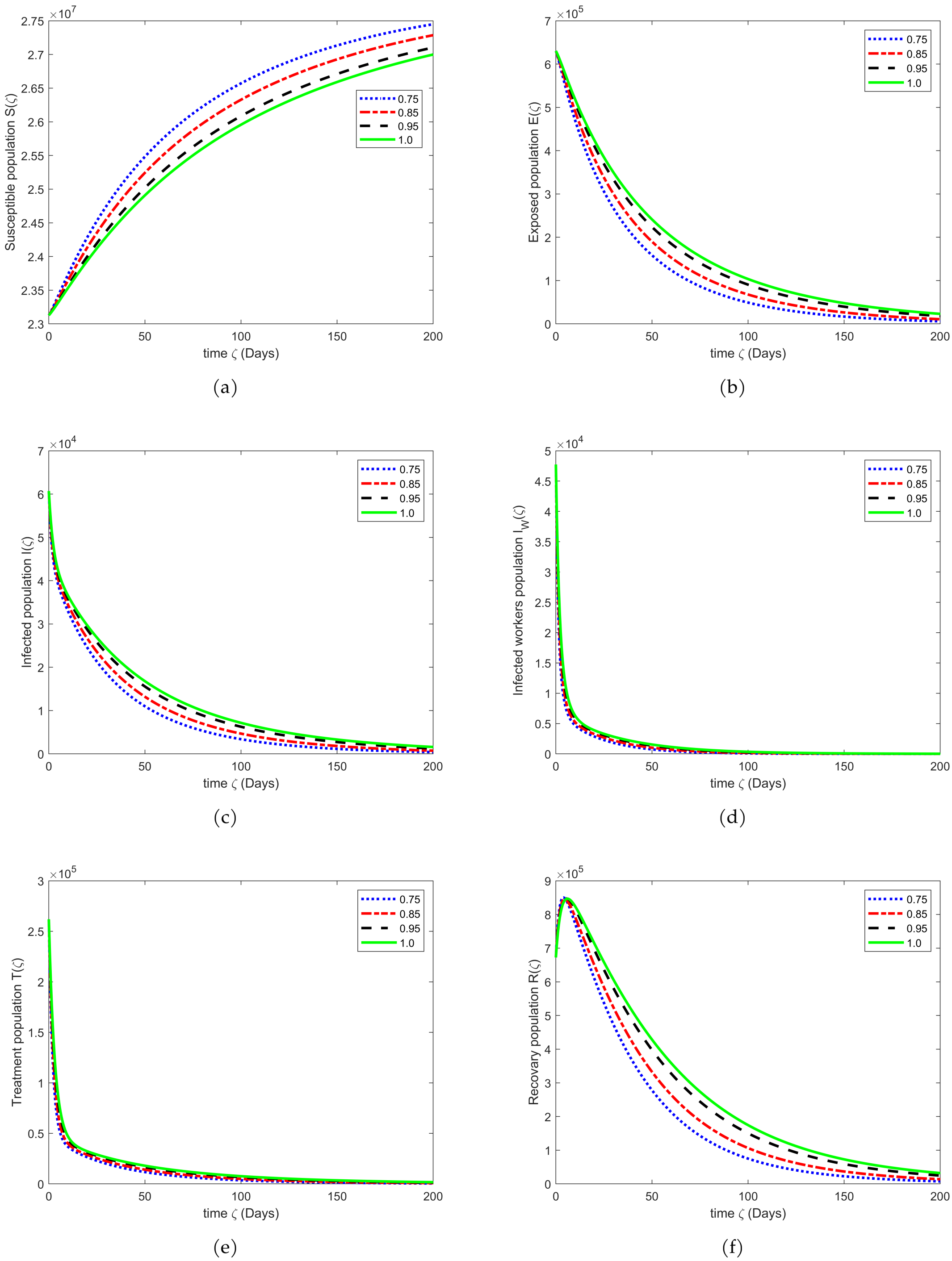

Figure 4: Graphical result of the epidemic Model (2) for various fractional orders $\eta = 1, 0.95, 0.85, 0.75$ via MEM method.

Further, we used the RK4 method to compare and check the validity of the Model (2) solutions. This is due to the MEM possessing second-order accuracy, while the RK4 method exhibits fourth-order accuracy [8]. Consequently, Figure 5 provides a more precise outcome. In summary, the integration of α and m parameters into the model has effectively reduced the transmission of TB.

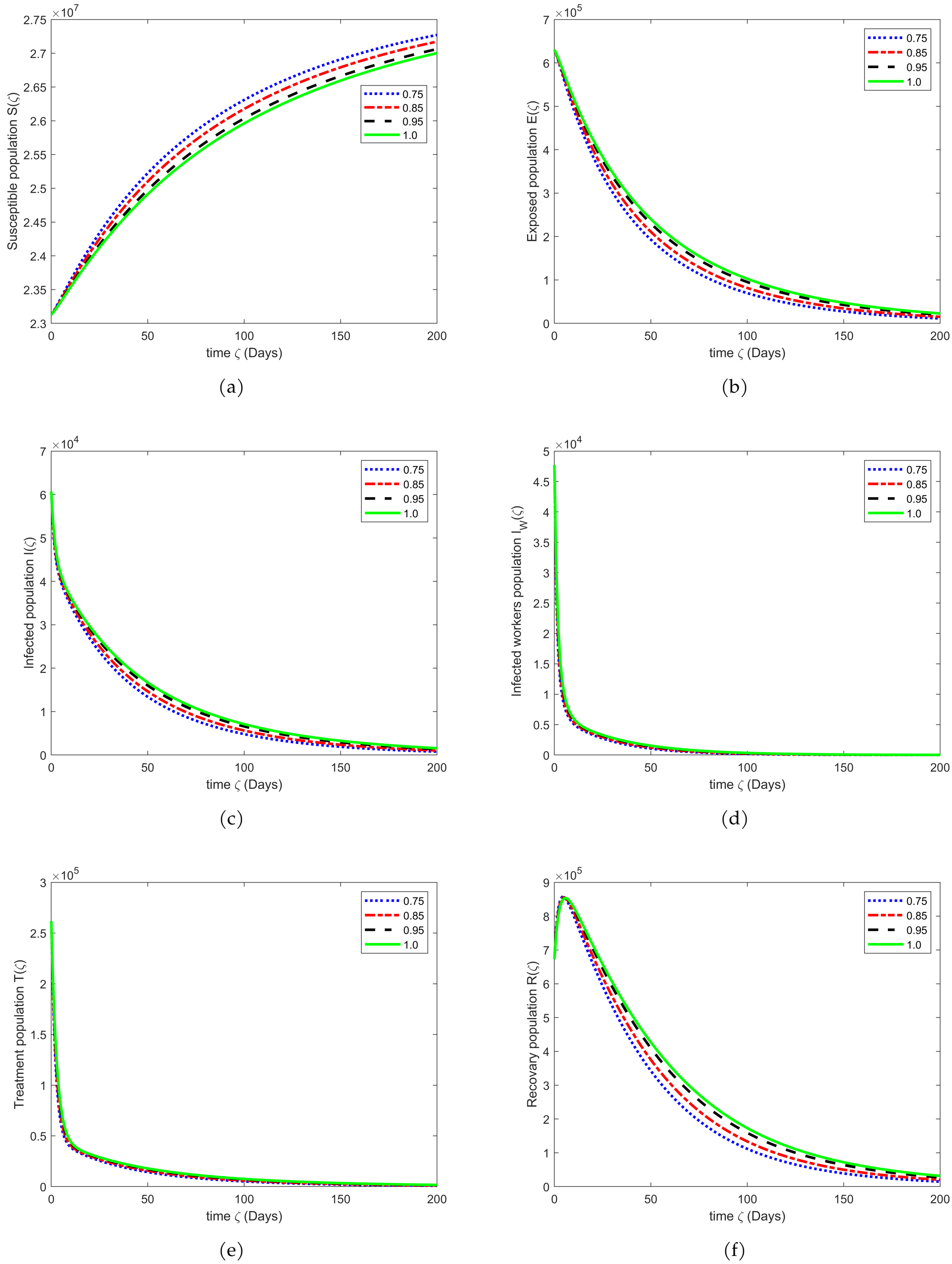

Figure 5: Graphical result of the epidemic Model (2) for various fractional orders $\eta = 1, 0.95, 0.85, 0.75$ via RK4 method.

8 Conclusions

Epidemiological models are essential for formulating strategies to comprehend, prevent, and manage infectious diseases via vaccination, social distancing, quarantine, and public awareness. The study provides an epidemiological model of TB transmission dynamics incorporating the effects of fractional-order derivatives and awareness. First, we proved the existence and uniqueness of the solution and examined the HU stability of the suggested model using fixed-point theory. Moreover, the sensitivity analysis was conducted to assess the relative impact of the media-influenced associated parameters. Since parameter α is so sensitive, TB may be completely eradicated by increasing the number of TB-infected migrants and seasonal farm workers seeking preventative treatment. Numerical simulations were performed using the MEM, and the results were visually interpreted for arbitrarily chosen parameters by MATLAB 2024 software. The accuracy and validity of these results were confirmed via comparison with the RK4 method. Graphical representations highlighted a significant reliance on parameter values, with varying outcomes observed for different values of η.

Notably, the value of $\mathbf{R_0} = 0.5473 < 1$, which indicates that the disease will become extinct entirely. This study's findings can contribute specific efforts to improve knowledge of TB through mass media and develop a positive attitude among seasonal farm workers and migrants towards the disease treatment. Furthermore, our findings encourage policy makers to develop and execute effective public health interventions for seasonal farm workers and migrants. Consequently, it is important to implement measures for seasonal farm workers and migrants to manage and eradicate TB. This comprehensive understanding offers a distinctive viewpoint on TB dynamics, enabling public health authorities and policymakers to develop more effective disease management and prevention techniques. In future work, we will utilize fractal-fractional derivative and optimal control theory in this research to examine the pandemics in TB infected regions.

Acknowledgement Authors are thankful to Prince Sultan University for APC and support through TAS research lab.

Conflicts of Interest The authors declare no conflict of interest.

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